THE MODULAR HEIGHT OF AN ABELIAN VARIETY AND ITS FINITENESS PROPERTY

ATSUSHI MORIWAKI

Abstract. In this note, we propose the modular height of an abelian variety defined over a field of finite type over $\mathbb{Q}$. Moreover, we prove its finiteness property.

Introduction

In the first proof of Mordell conjecture due to Faltings, the modular height of an abelian variety plays a crucial role. Especially, the finiteness property of the modular height is one of core parts for its proof. Almost every results over a number field (i.e., Tate conjecture, Shafarevich conjecture, Mordell conjecture and etc) has been generalized to a field of finite type over $\mathbb{Q}$. In this note, we propose the modular height of an abelian variety in general and prove its finiteness property.

Let K be a field of finite type over $\mathbb{Q}$. According to the paper [7], we need to fix a polarization of K in order to proceed with a theory of height functions over K, where a polarization of K is a pair $(B; \overline{H}_1, \ldots, \overline{H}_d)$ of a normal and integral projective scheme B over $\mathbb{Z}$ and a sequence of nef C^∞-hermitian line bundles $\overline{H}_1, \ldots, \overline{H}_d$ on B with the local ring of B at the generic point isomorphic to K. Here we assume that B is generically smooth, i.e., $B \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q}) \to \operatorname{Spec}(\mathbb{Q})$ is smooth.

Let A be an abelian variety over K. Then, using a Néron model of A over B in codimension one, we can introduce the Hodge sheaf $\lambda(A/K; B)$ of A which is a reflexive sheaf of rank one on B. Moreover, we can give a locally integrable hermitian metric $\| \cdot \|_{\mathrm{Fal}}$ of $\lambda(A/K; B)$ arising from the Faltings' metric of the good reduction part of the Néron model of A. Then,

$$\widehat{c}_1(\lambda(A/K; B), \| \cdot \|_{\mathrm{Fal}})$$

can be represented by a pair of a Weil divisor and a locally integrable function. Thus, we can define the modular height of A by the following formula:

$$h(A) = \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \cdot \widehat{c}_1(\lambda(A/K; B), \| \cdot \|_{\mathrm{Fal}})\right).$$

The main purpose of this note is to prove the following result (cf. Theorem 6.1):

> If $\overline{H}_1, \ldots, \overline{H}_d$ are big, then, for a fixed real number c, the set of isomorphism classes of abelian varieties A over K with $h(A) \leq c$ is finite.

Moret-Bailly [6] proved the geometric version of the above result using geometric intersection theory instead of Arakelov geometry. In this sense, the above is an arithmetic generalization of his result.

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1. Preliminaries

1.1. **Locally integrable hermitian metric.** Let M be a complex manifold and L a line bundle on M. Let $\|\cdot\|$ be a hermitian metric of L, that is, a collection of hermitian metrics of the stalks L_x at all $x \in X$. We say $\|\cdot\|$ is a *locally integrable hermitian metric* (or *L^1_{loc}-hermitian metric*) if, for each $x \in M$ and a local basis ω_x around x, $\log\|\omega_x\|$ is locally integrable around x. In other words, if $\|\cdot\|_0$ is a C^∞-hermitian metric of L, then $\log(\|\cdot\|/\|\cdot\|_0)$ is a locally integrable function on M.

Lemma 1.1.1. *Let M be a complex manifold and $(L, \|\cdot\|)$ a hermitian line bundle on M. Let s be a non-zero meromorphic section of L over M. Then, the hermitian metric $\|\cdot\|$ is locally integrable if and only if so is $\log\|s\|$.*

Proof. Let $\|\cdot\|_0$ be a C^∞-hermitian metric of L. Then,

$$\log\|s\| = \log(\|\cdot\|/\|\cdot\|_0) + \log\|s\|_0.$$

Note that $\log\|s\|_0$ is locally integrable. Thus, $\log\|s\|$ is locally integrable if and only if so is $\log(\|\cdot\|/\|\cdot\|_0)$. □

Lemma 1.1.2. *Let $f : Y \to X$ be a surjective, proper and generically finite morphism of non-singular varieties over $\mathbb{C}$. Let $(L, \|\cdot\|)$ be a hermitian line bundle on X. Assume that there are a non-empty Zariski open set U of X and a hermitian line bundle $(L', \|\cdot\|')$ on Y such that $(L', \|\cdot\|')$ is isometric to $f^*(L, \|\cdot\|)$ over $f^{-1}(U)$. If $\|\cdot\|'$ is locally integrable, then so is $\|\cdot\|$.*

Proof. Shrinking U if necessarily, we may assume that f is étale over U. We set $V = f^{-1}(U)$. Let s be a non-zero rational section of L. Note that there is a divisor D on Y such that $L' = f^*(L) \otimes \mathcal{O}_Y(D)$ and $\mathrm{Supp}(D) \subseteq Y \setminus V$. Thus, $f^*(s)$ gives rise to a rational section s' of L'. Then, $\log\|s'\|'$ is locally integrable by Lemma 1.1.1. Since $f^*(\log\|s\|)|_V = \log\|s'\|'|_V$, we can see that $f^*(\log\|s\|)$ is locally integrable. Let $[f^*(\log\|s\|)]$ be a current associated to the locally integrable function $f^*(\log\|s\|)$. Then, [4, Proposition 1.2.5], there is a locally integrable function g on X with $f_*[f^*(\log\|s\|)] = [g]$. Since f is étale over U, we can easily see that

$$(f|_V)_*[(f|_V)^*(\log\|s\||_U)] = \deg(f)[\log\|s\||_U].$$

Thus, $g = \deg(f)\log\|s\|$ almost everywhere over U. Therefore, so is over X because U is a non-empty Zariski open set of X. Hence, $\log\|s\|$ is locally integrable on X. □

1.2. **Hermitian metric with logarithmic singularities.** Let X be a normal variety over $\mathbb{C}$ and Y a proper closed subscheme of X. Let $(L, \|\cdot\|)$ be a hermitian line bundle on X. We say $(L, \|\cdot\|)$ is a *C^∞-hermitian line bundle with logarithmic singularities along* Y if the following conditions are satisfied:

(1) $\|\cdot\|$ is C^∞ over $X \setminus Y$.
(2) Let $\|\cdot\|_0$ be a C^∞-hermitian metric of L. For each $x \in Y$, let $f_1, \ldots, f_m$ be a system of local equations of Y around x, i.e., Y is given by $\{z \in X \mid f_1(z) = \cdots = f_m(z) = 0\}$ around x. Then, there are positive constants C and r such that

$$\max\left\{\frac{\|\cdot\|}{\|\cdot\|_0}, \frac{\|\cdot\|_0}{\|\cdot\|}\right\} \leq C\left(-\sum_{i=1}^{m}\log|f_i|\right)^r$$

around x.

Note that the above definition does not depend on the choice of the system of local equations $f_1, \ldots, f_m$. Moreover, it is easy to see that if $(L, \|\cdot\|)$ is a C^∞-hermitian line bundle with logarithmic singularities along Y, then $\|\cdot\|$ is locally integrable.

Lemma 1.2.1. *Let $\pi : X' \to X$ be a proper morphism of normal varieties over $\mathbb{C}$ and Y a proper closed subscheme of X. Let $(L, \|\cdot\|)$ be a hermitian line bundle on X such that $\|\cdot\|$ is C^∞ over $X \setminus Y$. If $\pi(X') \not\subseteq Y$ and $(L, \|\cdot\|)$ has logarithmic singularities along Y, then so does $\pi^*(L, \|\cdot\|)$ along $\pi^{-1}(Y)$. Moreover, if π is surjective and $\pi^*(L, \|\cdot\|)$ has logarithmic singularities along $\pi^{-1}(Y)$, then so does $(L, \|\cdot\|)$ along Y.*

Proof. Let $\{f_1, \ldots, f_m\}$ be a system of local equations of Y. Then, $\{\pi^*(f_1), \ldots, \pi^*(f_m)\}$ is a system of local equation of $\pi^{-1}(Y)$. Thus, our assertion is obvious. $\square$

1.3. **Faltings' metric.** Let X be a normal variety over $\mathbb{C}$. Let $f : A \to X$ be a g-dimensional semi-abelian scheme over X. We assume that there is a non-empty Zariski open set U of X such that f is an abelian scheme over U. Let $\lambda_{A/X}$ be the Hodge line bundle of $A \to X$, i.e.,

$$\lambda_{A/X} = \det\left(\epsilon^*\left(\Omega_{A/X}\right)\right),$$

where $\epsilon : X \to A$ is the identity of the semi-abelian scheme $A \to X$. At each $x \in U$, we can give a hermitian metric of $(\lambda_{A/X})_x$ in the following way: For $\alpha \in \bigwedge^g H^0(\Omega_{A_x})$,

$$(\|\alpha\|_x)^2 = \left(\frac{\sqrt{-1}}{2}\right)^g \int_{A_x} \alpha \wedge \bar{\alpha}.$$

Then, a collection of metrics $\{\|\cdot\|_x\}_{x\in U}$ gives rise to a C^∞-hermitian metrics $\|\cdot\|_{\mathrm{Fal}}$ of $\lambda_{A/X}\big|_U$. Moreover, it is well-known that $\|\cdot\|_{\mathrm{Fal}}$ extends to a C^∞-hermitian metric of $\lambda_{A/X}$ with logarithmic singularities along $X \setminus U$ (cf. [10, Théorèm 3.2 in Exposé I]). By abuse of notation, this extended metric is also denoted by $\|\cdot\|_{\mathrm{Fal}}$ and is called Faltings' metric of $\lambda_{A/X}$.

Lemma 1.3.1. *Let X be a smooth variety over $\mathbb{C}$ and X_0 a non-empty Zariski open set of X. Let $A_0 \to X_0$ be an abelian scheme over X_0. Let λ be a line bundle on X such that $\lambda|_{X_0}$ gives rise to the Hodge line bundle λ_{A_0/X_0} of $A_0 \to X_0$. Then, Faltings' metric $\|\cdot\|_{\mathrm{Fal}}$ of λ_{A_0/X_0} over X_0 extends to a locally integrable metric of λ over X.*

Proof. By virtue of Lemma 1.5.2 (Gabber's lemma), there is a proper, surjective and generically finite morphism $\pi : X' \to X$ of smooth varieties over $\mathbb{C}$ such that the abelian scheme $A_0 \times_{X_0} \pi^{-1}(X_0)$ over $\pi^{-1}(X_0)$ extends to a semi-abelian scheme $f' : A' \to X'$. Let $\lambda_{A'/X'}$ be the Hodge line bundle of $A' \to X'$ and $\|\cdot\|'_{\mathrm{Fal}}$ Faltings' metric of $\lambda_{A'/X'}$. Then, $(\lambda_{A'/X'}, \|\cdot\|'_{\mathrm{Fal}})\big|_{X'_0}$ is isometric to $\pi_0^*(\lambda_{A_0/X_0}, \|\cdot\|_{\mathrm{Fal}})$, where $X'_0 = \pi^{-1}(X_0)$ and $\pi_0 = \pi|_{X'_0}$. Therefore, by Lemma 1.1.2, $\|\cdot\|_{\mathrm{Fal}}$ extends to a locally integrable metric over X. $\square$

1.4. **Néron model.** Let R be a discrete valuation ring and K the quotient field of R. Let A be an abelian variety over K. Then, there is a smooth group scheme $\mathcal{A} \to \mathrm{Spec}(R)$ of finite type over R with the following properties (cf. [1]):

(1) The generic fiber of $\mathcal{A} \to \mathrm{Spec}(R)$ is A.

(2) Let $\mathcal{X} \to \operatorname{Spec}(R)$ be a smooth scheme over R and X the generic fiber of $\mathcal{X} \to \operatorname{Spec}(R)$. Then, any morphism $X \to A$ over K extends uniquely to a morphism $\mathcal{X} \to \mathcal{A}$ over R.

The smooth group scheme $\mathcal{A} \to \operatorname{Spec}(R)$ is called the Néron model of A over R. We would like to generalize it to a higher dimensional base scheme.

Let B be an irreducible noetherian normal scheme and K the function field of B, i.e., the local ring at the generic point of B. Let A be an abelian variety over K. A smooth group scheme $f : \mathcal{A} \to B$ is called the Néron model of A over B if (1) $f : \mathcal{A} \to B$ is of finite type over B and (2) for every point $x \in B$ of codimension one, $\mathcal{A}|_{\operatorname{Spec}(\mathcal{O}_x)} \to \operatorname{Spec}(\mathcal{O}_x)$ is the Néron model of A over $\operatorname{Spec}(\mathcal{O}_x)$. Let $\mathcal{X} \to B$ a smooth scheme over B and X the generic fiber of $\mathcal{X} \to B$. Let $\phi_K : X \to A$ be a morphism over K. If $f : \mathcal{A} \to B$ is the Néron model of A, then, by the property (2) and Weil's extension theorem (cf. [1, Theorem 1 in 4.4]), there is the unique extension $\phi : \mathcal{X} \to \mathcal{A}$ of ϕ_K over B.

Proposition 1.4.1. *Let B, K and A be same as above. Then there is a non-empty big open set B' of B (i.e., $\operatorname{codim}(B \setminus B') \geq 2$) such that a Néron model of A over B' exists. This Néron model is called a Néron model of A over B in codimension one.*

Proof. First of all, we can take a non-empty Zariski open set B_0 of B and an abelian scheme $\mathcal{A}_0 \to B_0$ whose generic fiber is A. Let

$$B \setminus B_0 = D_1 \cup D_2 \cup \cdots \cup D_n$$

be the irreducible decomposition of $B \setminus B_0$. We assume that $\operatorname{codim}(D_i) = 1$ for $1 \leq i \leq r$ and $\operatorname{codim}(D_j) \geq 2$ for $r < j \leq n$. Let x_i be the generic point of D_i. For $1 \leq i \leq r$, let $A_i \to \operatorname{Spec}(\mathcal{O}_{x_i})$ be the Néron model of A over $\operatorname{Spec}(\mathcal{O}_{x_i})$. Then, there are an open set B_i containing x_i and a smooth group scheme $\mathcal{A}_i \to B_i$ as the extension of $A_i \to \operatorname{Spec}(\mathcal{O}_{x_i})$ such that $\mathcal{A}_i$ is of finite type over B_i. Replacing B_i by $B_i \setminus (D_1 \cup \cdots \cup D_{i-1} \cup D_{i+1} \cup \cdots \cup D_n)$, we may assume that

$$B_i \cap (D_1 \cup \cdots \cup D_{i-1} \cup D_{i+1} \cup \cdots \cup D_n) = \emptyset.$$

By [2, Lemma 3.3 in Chapter I] or [3, Proposition 2.7 in Chapter 1], $\mathcal{A}_0 \to B_0$ coincides with $\mathcal{A}_i \to B_i$ over $B_0 \cap B_i$. Moreover, $B_i \cap B_j \subseteq B_0$ for $1 \leq i \neq j \leq r$. Therefore, if we set $B' = B_0 \cup B_1 \cup \cdots \cup B_r$, we have our desired smooth group scheme $\mathcal{A} \to B'$. $\square$

1.5. **Semi-abelian reduction.** Let B be an irreducible normal noetherian scheme and K the local ring at the generic point of B. Let A be an abelian variety over K. We say A has semi-abelian reduction over B in codimension one if there are a big open set B_1 of B (i.e., $\operatorname{codim}(B \setminus B') \geq 2$) and a semi-abelian scheme $\mathcal{A} \to B_1$ such that the generic fiber of $\mathcal{A} \to B_1$ is A.

Proposition 1.5.1. *Let B, K and A be same as above. Let m be a positive integer which has a factorization $m = m_1 m_2$ with $m_1, m_2 \geq 3$ and m_1 and m_2 relatively prime (for example $m = 12 = 3 \cdot 4$). If $A[m](\overline{K}) \subseteq A(K)$, then A has semi-abelian reduction in codimension one over B.*

Proof. Let x be a codimension one point of B. Then, there is m_i which is not divisible by the characteristic of the residue field of $\mathcal{O}_{B,x}$. Moreover, $A[m_i](\overline{K}) \subseteq A(K)$. Thus, by [9, exposé 1, Corollaire 5.18], A has semi-abelian reduction at x.

Let B_0 be a non-empty Zariski open set of B such that we can take an abelian scheme $\mathcal{A}_0 \to B_0$ whose generic fiber is A. Let

$$B \setminus B_0 = D_1 \cup D_2 \cup \cdots \cup D_n$$

be the irreducible decomposition of $B \setminus B_0$. We assume that $\operatorname{codim}(D_i) = 1$ for $1 \leq i \leq r$ and $\operatorname{codim}(D_j) \geq 2$ for $r < j \leq n$. Let x_i be the generic point of D_i. Then, for each $i = 1, \ldots, r$, there are an open set B_i of B and a semi-abelian scheme $\mathcal{A}_i \to B_i$ with $x_i \in B_i$. Shrinking B_i if necessarily, we may assume that $B_i \cap B_j \subseteq B_0$ for all $1 \leq i \neq j \leq r$. Thus, as in Proposition 1.4.1, if we set $B' = B_0 \cup B_1 \cup \cdots \cup B_r$, then we have our desired semi-abelian scheme $\mathcal{A} \to B'$. □

Lemma 1.5.2 (Gabber's lemma)**.** *Let U be a dense Zariski open set of an integral, normal and excellent scheme S and A an abelian scheme over U. Then, there is a proper, surjective and generically finite morphism $\pi : S' \to S$ of integral, normal and excellent schemes such that the abelian scheme $A \times_U f^{-1}(U)$ over $f^{-1}(U)$ extends to an semi-abelian scheme over S'*

Proof. In [10, Théorèm and Proposition 4.10 in Exposé V], the existence of $\pi : S' \to S$ and the extension of the abelian scheme is proved under the assumption $\pi : S' \to S$ is proper and surjective. Let S'_η be the generic fiber of π. Let z be the closed point of S'_η and Z the closure of z in S'. Moreover, let S_1 be the normalization of Z. Then, $\pi_1 : S_1 \to Z \to S$ is our desired morphism. □

1.6. **The Hodge sheaf of an abelian variety.** Let $G \to S$ be a smooth group scheme over S. Then, *the Hodge line bundle $\lambda_{G/S}$ of $G \to S$* is given by

$$\lambda_{G/S} = \det\left(\epsilon^*\left(\Omega_{G/S}\right)\right),$$

where ϵ is the identity of the group scheme $G \to S$.

Let B be an irreducible and normal noetherian scheme. Let K be the function field of B (i.e., the local ring at the generic point). Let A be an abelian variety over K. By Proposition 1.4.1, there is a big open set B' of B such that the Néron model $\mathcal{A}' \to B'$ of A over B' exists. Let $\iota : B' \to B$ be the natural inclusion map. *The Hodge sheaf $\lambda(A/K;B)$ of A with respect to B* is defined by

$$\lambda(A/K;B) = \iota_*\left(\lambda_{\mathcal{A}'/B'}\right).$$

Note that $\lambda(A/K;B)$ is a reflexive sheaf of rank one on B.

From now on, we assume that the characteristic of K is zero. Let $\phi : A \to A'$ be an isogeny of abelian varieties over K. Let $\mathcal{A}$ and $\mathcal{A}'$ be the Néron models in codimension one over B of A and A' respectively. Since there is an injective homomorphism

$$\phi^* : \lambda(A'/K;B) \to \lambda(A/K;B),$$

we can find an effective Weil divisor D_ϕ such that

$$c_1(\lambda(A'/K;B)) + D_\phi = c_1(\lambda(A/K;B)).$$

The ideal sheaf $\mathcal{O}_B(-D_\phi)$ is denoted by $\mathcal{I}_\phi$.

Lemma 1.6.3. *Let $\phi^\vee : A'^\vee \to A^\vee$ be the dual of $\phi : A \to A'$. We assume that $B = \operatorname{Spec}(R)$ for some discrete valuation ring R and that A and A' have semi-abelian reduction over R. Then, $\mathcal{I}_\phi \cdot \mathcal{I}_{\phi^\vee} = \deg(\phi)R$.*

Proof. Let R' be an extension of R such that R' is a complete discrete valuation ring and the residue field of R' is algebraically closed. Then, by [10, Exposé VII, Théroèm 2.1.1], $(\mathcal{I}_\phi \cdot \mathcal{I}_{\phi^\vee})R' = \deg(\phi)R'$. Here R' is faithfully flat over R. Thus, $\mathcal{I}_\phi \cdot \mathcal{I}_{\phi^\vee} = \deg(\phi)R$. $\square$

1.7. **The moduli of abelian varieties.** To prove the finiteness property of the modular height, it is very important to get a good compactification of the moduli space of abelian varieties. For simplicity, an abelian variety with a polarization of degree l^2 is called an *l-polarized abelian variety.*

Theorem 1.7.1. *Let g, l and m be positive integers with $m \geq 3$. Let $\mathbb{A}_{g,l,m,\mathbb{Q}}$ be the moduli space of g-dimensional and l-polarized abelian varieties over $\mathbb{Q}$ with an m-level structure. Then, there exists* (a) *normal projective arithmetic varieties $\mathbb{A}^*_{g,l,m}$ and Y^* (i.e., $\mathbb{A}^*_{g,l,m}$ and Y^* are normal and integral schemes flat and projective over $\mathbb{Z}$),* (b) *a surjective and generically finite morphism $f : Y^* \to \mathbb{A}^*_{g,l,m}$,* (c) *a positive integer n,* (d) *a line bundle L on $\mathbb{A}^*_{g,l,m}$, and* (e) *a semi-abelian scheme $G \to Y^*$ with the following properties:*

(1) *$\mathbb{A}_{g,l,m,\mathbb{Q}}$ is a Zariski open set of $\mathbb{A}^*_{g,l,m,\mathbb{Q}} = \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$ and L is very ample on $\mathbb{A}^*_{g,l,m}$.*
(2) *Let λ_{G/Y^*} be the Hodge line bundle of the semi-abelian scheme $G \to Y^*$. Then, $f^*(L) = \lambda^{\otimes n}_{G/Y^*}$ on $Y^*_{\mathbb{Q}} = Y^* \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$.*
(3) *Let $U_{\mathbb{Q}} \to \mathbb{A}_{g,l,m,\mathbb{Q}}$ be the universal g-dimensional and l-principally polarized abelian scheme with an m-level structure. Let $Y_{\mathbb{Q}}$ be the pull-back of $\mathbb{A}_{g,l,m,\mathbb{Q}}$ by $f_{\mathbb{Q}} : Y^*_{\mathbb{Q}} \to \mathbb{A}^*_{g,l,m,\mathbb{Q}}$, i.e., $Y_{\mathbb{Q}} = (f_{\mathbb{Q}})^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}})$. Then, $G_{\mathbb{Q}} \to Y^*_{\mathbb{Q}}$ is an extension of the abelian scheme $U_{\mathbb{Q}} \times_{\mathbb{A}_{g,l,m,\mathbb{Q}}} Y_{\mathbb{Q}} \to Y_{\mathbb{Q}}$. (Note that $G|_{Y_{\mathbb{Q}}} \to Y_{\mathbb{Q}}$ is naturally a g-dimensional and l-polarized abelian scheme with an m-level structure.)*
(4) *L has a metric $\|\cdot\|$ over $\mathbb{A}_{g,l,m,\mathbb{Q}}(\mathbb{C})$ such that $f^*((L, \|\cdot\|))$ is isometric to $\left(\lambda_{G/Y^*}, \|\cdot\|_{\mathrm{Fal}}\right)^{\otimes n}$ over $Y_{\mathbb{Q}}(\mathbb{C})$. Moreover, $\|\cdot\|$ has logarithmic singularities along $\mathbb{A}^*_{g,l,m,\mathbb{Q}}(\mathbb{C}) \setminus \mathbb{A}_{g,l,m,\mathbb{Q}}(\mathbb{C})$.*

Proof. Let $U_{\mathbb{Q}} \to \mathbb{A}_{g,l,m,\mathbb{Q}}$ be the universal l-polarized abelian scheme with an m-level structure. By [10, Théorème 2.2 in Exposé IV], there are a normal projective variety $\mathbb{A}^*_{g,l,m,\mathbb{Q}}$, a positive integer n and a very ample line bundle $L_{\mathbb{Q}}$ on $\mathbb{A}^*_{g,l,m,\mathbb{Q}}$ with the following properties:

(i) $\mathbb{A}_{g,l,m,\mathbb{Q}}$ is an Zariski open set of $\mathbb{A}^*_{g,l,m,\mathbb{Q}}$.
(ii) By Gabber's lemma (cf. Lemma 1.5.2), there is a surjective and generically finite morphism $h_{\mathbb{Q}} : S'_{\mathbb{Q}} \to \mathbb{A}^*_{g,l,m,\mathbb{Q}}$ of normal projective varieties over $\mathbb{Q}$ such that the abelian scheme $U_{\mathbb{Q}} \times_{\mathbb{A}_{g,l,m,\mathbb{Q}}} h_{\mathbb{Q}}^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}}) \to h_{\mathbb{Q}}^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}})$ extends to a semi-abelian scheme $G'_{\mathbb{Q}} \to S'_{\mathbb{Q}}$. Then, $h^*_{\mathbb{Q}}(L_{\mathbb{Q}}) = \lambda^{\otimes n}_{G'_{\mathbb{Q}}/S'_{\mathbb{Q}}}$.

Since $L_{\mathbb{Q}}$ is very ample, there is an embedding $\mathbb{A}^*_{g,l,m,\mathbb{Q}} \hookrightarrow \mathbb{P}^N_{\mathbb{Q}}$ in terms of $L_{\mathbb{Q}}$. Let $\mathbb{A}^*_{g,l,m}$ be the closure of the image of

$$\mathbb{A}^*_{g,l,m,\mathbb{Q}} \hookrightarrow \mathbb{P}^N_{\mathbb{Q}} \to \mathbb{P}^N_{\mathbb{Z}}.$$

Moreover, let L be the pull-back of $\mathcal{O}_{\mathbb{P}^N_{\mathbb{Z}}}(1)$ by the embedding $\mathbb{A}^*_{g,l,m} \hookrightarrow \mathbb{P}^N_{\mathbb{Z}}$. Then, $\mathbb{A}^*_{g,l,m,\mathbb{Q}} = \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$ and $L_{\mathbb{Q}} = L|_{\mathbb{A}^*_{g,l,m,\mathbb{Q}}}$. Let S' be the normalization of $\mathbb{A}^*_{g,l,m}$ in the function field of $S'_{\mathbb{Q}}$. Then, there is an open set S'_0 of S' such that G' is an ableian scheme over S'_0 and $G' \times_{S'} S'_0 \to S'_0$ coincides with the abelian scheme $U_{\mathbb{Q}} \times_{\mathbb{A}_{g,l,m,\mathbb{Q}}} h_{\mathbb{Q}}^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}}) \to h_{\mathbb{Q}}^{-1}(\mathbb{A}_{g,l,m,\mathbb{Q}})$ over $\mathbb{Q}$. Thus, using Gabber's lemma again, there are a surjective and generically finite morphism of normal arithmetic varieties $h_2 : Y^* \to S'$ and a semi-abelian scheme $G \to Y^*$ such that $G \to Y^*$ is an extension of $G' \times_{S'} h_2^{-1}(S'_0) \to h_2^{-1}(S'_0)$. We set $Y^*_{\mathbb{Q}} = Y^* \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$. Then, G over $Y^*_{\mathbb{Q}}$ is equal to $G'_{\mathbb{Q}} \times_{S'_{\mathbb{Q}}} Y^*_{\mathbb{Q}} \to Y^*_{\mathbb{Q}}$ by the uniqueness of semi-abelian extension. Thus, if we set $f = h \cdot h_1$, then $f^*(L) = \lambda_{G/Y^*}^{\otimes n}$ over $Y^*_{\mathbb{Q}}$.

Finally, since $L_{\mathbb{Q}}|_{\mathbb{A}_{g,l,m,\mathbb{Q}}} = \lambda^{\otimes n}_{U_{\mathbb{Q}}/\mathbb{A}_{g,l,m,\mathbb{Q}}}$, if we give $L_{\mathbb{Q}}$ a metric arising from the Faltings' metric of $\lambda_{U_{\mathbb{Q}}/\mathbb{A}_{g,l,m,\mathbb{Q}}}$, then assertion of (4) follows from Lemma 1.2.1 and [10, Théorèm 3.2 in Exposé I]. $\square$

1.8. **Arakelov geometry.** In this note, a flat and quasi-projective integral scheme over $\mathbb{Z}$ is called an *arithmetic variety*. If it is smooth over $\mathbb{Q}$, then it is said to be *generically smooth*.

Let X be a generically smooth arithmetic variety. A pair (Z, g) is called an *arithmetic cycle of codimension* p if Z is a cycle of codimension p and g is a current of type $(p-1, p-1)$ on $X(\mathbb{C})$. We denote by $\widehat{Z}^p(X)$ the set of all arithmetic cycles on X. We set

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X)/\sim,$$

where $\sim$ is the arithmetic linear equivalence.

Let $\overline{L} = (L, \|\cdot\|)$ be a C^∞-hermitian line bundle on X. Then, a homomorphism

$$\widehat{c}_1(\overline{L})\cdot : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p+1}(X)$$

is define by

$$\widehat{c}_1(\overline{L}) \cdot (Z, g) = \left(\operatorname{div}(s) \text{ on } Z, [-\log(\|s\|_Z^2)] + c_1(\overline{L}) \wedge g\right),$$

where s is a rational section of $L|_Z$ and $[-\log(\|s\|_Z^2)]$ is a current given by $\phi \mapsto -\int_{Z(\mathbb{C})} \log(\|s\|_Z^2)\phi$.

Here we assume that X is projective. Then we can define the arithmetic degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{\dim X}(X) \to \mathbb{R}$$

by

$$\widehat{\deg}\left(\sum_P n_P P, g\right) = \sum_P n_P \log(\#(\kappa(P))) + \frac{1}{2}\int_{X(\mathbb{C})} g.$$

Thus, if C^∞-hermitian line bundles $\overline{L}_1, \ldots, \overline{L}_{\dim X}$ are given, then we can get the number

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_{\dim X})\right),$$

which is called the *arithmetic intersection number of* $\overline{L}_1, \ldots, \overline{L}_{\dim X}$.

Let X be a projective arithmetic variety. Note that X is not necessarily generically smooth. Let $\overline{L}_1, \ldots, \overline{L}_{\dim X}$ be C^∞-hermitian line bundles on X. By [5], we can find a generic resolution of singularities $\mu : Y \to X$, i.e., $\mu : Y \to X$ is a projective and birational

morphism such that Y is a generically smooth projective arithmetic variety. Then, we can see that the arithmetic intersection number

$$\widehat{\deg}\left(\widehat{c}_1(\mu^*(\overline{L}_1))\cdots\widehat{c}_1(\mu^*(\overline{L}_{\dim X}))\right)$$

does not depend on the choice of the generic resolution of singularities $\mu : Y \to X$. Thus, we denote this number by

$$\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_{\dim X})\right).$$

Let $\overline{L}_1, \ldots, \overline{L}_l$ be C^∞-hermitian line bundles on a projective arithmetic variety X. Let V be an l-dimensional integral closed subscheme on X. Then, $\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_l) \mid V\right)$ is defined by

$$\widehat{\deg}\left(\widehat{c}_1(\left.\overline{L}_1\right|_V)\cdots\widehat{c}_1(\left.\overline{L}_l\right|_V)\right).$$

Moreover, for an l-dimensional cycle $Z = \sum_i n_i V_i$ on X, $\widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_l) \mid Z\right)$ is defined by

$$\sum_i n_i \widehat{\deg}\left(\widehat{c}_1(\overline{L}_1)\cdots\widehat{c}_1(\overline{L}_l) \mid V_i\right).$$

1.9. **The positivity of C^∞-hermitian $\mathbb{Q}$-line bundles on a projective arithmetic variety.** Let X be a projective arithmetic variety and $\overline{L}$ a C^∞-hermitian $\mathbb{Q}$-line bundle on X. Let us introduce several kinds of the positivity of C^∞-hermitian $\mathbb{Q}$-line bundles.

•**ample**: We say $\overline{L}$ is *ample* if L is ample on X, $c_1(\overline{L})$ is positive form on $X(\mathbb{C})$, and there is a positive number n such that $L^{\otimes n}$ is generated by the set $\{s \in H^0(X, L^{\otimes n}) \mid \|s\|_{\sup} < 1\}$.

•**nef**: We say $\overline{L}$ is *nef* if $c_1(\overline{L})$ is a semipositive form on $X(\mathbb{C})$ and, for all one-dimensional integral closed subschemes Γ of X, $\widehat{\deg}\left(\widehat{c}_1(\overline{L}) \mid \Gamma\right) \geq 0$.

•**big**: $\overline{L}$ is said to be *big* if $\operatorname{rk}_{\mathbb{Z}} H^0(X, L^{\otimes m}) = O(m^{\dim X_{\mathbb{Q}}})$ and there is a non-zero section s of $H^0(X, L^{\otimes n})$ with $\|s\|_{\sup} < 1$ for some positive integer n.

•**$\mathbb{Q}$-effective**: $\overline{L}$ is said to be *$\mathbb{Q}$-effective* if there is a positive integer n and a non-zero $s \in H^0(X, L^{\otimes n})$ with $\|s\|_{\sup} \leq 1$.

•**pseudo-effective**: $\overline{L}$ is said to be *pseudo-effective* if there are (1) a sequence $\{\overline{L}_n\}_{n=1}^\infty$ of $\mathbb{Q}$-effective C^∞-hermitian $\mathbb{Q}$-line bundles, (2) C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{E}_1, \ldots, \overline{E}_r$ and (3) sequences $\{a_{1,n}\}_{n=1}^\infty, \ldots, \{a_{r,n}\}_{n=1}^\infty$ of rational numbers such that

$$\widehat{c}_1(\overline{L}) = \widehat{c}_1(\overline{L}_n) + \sum_{i=1}^r a_{i,n}\widehat{c}_1(\overline{E}_i)$$

in $\widehat{\mathrm{CH}}^1(X) \otimes \mathbb{Q}$ and $\lim_{n\to\infty} a_{i,n} = 0$ for all i. If $\overline{L}_1 \otimes \overline{L}_2^{\otimes -1}$ is pseudo-effective for C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{L}_1, \overline{L}_2$ on X, then we denote this by $\overline{L}_1 \succsim \overline{L}_2$.

1.10. **Polarization of a finitely generated field over $\mathbb{Q}$.** Let K be a field of finite type over the rational number field $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$. A pair $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ of a normal projective arithmetic variety B and a sequence $\overline{H}_1, \ldots, \overline{H}_d$ of C^∞-hermitian line bundles on B is called a polarization if the function field of B (i.e., the local ring at the generic point) is K and $\overline{H}_1, \ldots, \overline{H}_d$ are nef. Here $\deg(\overline{B})$ is given by

$$\int_{B(\mathbb{C})} c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_d).$$

Namely,

$$\deg(\overline{B}) = \begin{cases} [K:\mathbb{Q}] & \text{if } d = 0, \\ \deg((H_1)_{\mathbb{Q}} \cdots (H_d)_{\mathbb{Q}}) \text{ on } B \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q}) & \text{if } d > 0. \end{cases}$$

If B is generically smooth, then the polarization $\overline{B}$ is said to be *generically smooth*. Moreover, we say the polarization $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ is *fine* (resp. *strictly fine*) if there is a generically finite morphism $\pi : B' \to B$ of normal projective arithmetic varieties, a generically finite morphism $\mu : B' \to (\mathbb{P}^1_{\mathbb{Z}})^d$ and ample C^∞-hermitian $\mathbb{Q}$-line bundles $\overline{L}_1, \ldots, \overline{L}_d$ on $\mathbb{P}^1_{\mathbb{Z}}$ such that $\pi^*(\overline{H}_i) \otimes \mu^*(p_i^*(\overline{L}_i))^{\otimes -1}$ is pseudo-effective (resp. $\mathbb{Q}$-effective) for every i, where $p_i : (\mathbb{P}^1_{\mathbb{Z}})^d \to \mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor. Note that if $\overline{H}_1, \ldots, \overline{H}_d$ are big, then the polarization $(B; \overline{H}_1, \ldots, \overline{H}_d)$ is strictly fine. Moreover, if $\overline{B}$ is fine, then $\deg(\overline{B}) > 0$.

Let us see the following proposition.

Proposition 1.10.1. *Let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a strictly fine polarization of K. Then, for all h, the number of prime divisors Γ on B with*

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_1) \mid \Gamma) \leq h$$

is finite.

Proof. Let us begin with the following lemma.

Lemma 1.10.2. *Let $\pi : X' \to X$ be a generically finite morphism of normal projective arithmetic varieties. Let $\overline{H}_1, \ldots, \overline{H}_d$ be nef C^∞-hermitian line bundles on X, where $d = \dim X_{\mathbb{Q}}$. Then, the following are equivalent:*

(1) *For all h, the number of prime divisors Γ on X with*

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_1) \mid \Gamma) \leq h$$

is finite

(2) *For all h', the number of prime divisors Γ' on X' with*

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1)) \cdots \widehat{c}_1(\pi^*(\overline{H}_1)) \mid \Gamma') \leq h'$$

is finite.

Proof. Let X_0 be the maximal Zariski open set of X such that X_0 is regular and π is finite over X_0. Then, $\operatorname{codim}(X \setminus X_0) \geq 2$. We set $X'_0 = \pi^{-1}(X_0)$ and $\pi_0 = \pi|_{X'_0}$. Let $\operatorname{Div}(X)$ and $\operatorname{Div}(X')$ be the groups of Weil divisors on X and X' respectively. Then, a homomorphism $\pi^\star : \operatorname{Div}(X) \to \operatorname{Div}(X')$ is defined by the compositions of homomorphisms:

$$\operatorname{Div}(X) \to \operatorname{Div}(X_0) \xrightarrow{\pi_0^*} \operatorname{Div}(X'_0) \to \operatorname{Div}(X'),$$

where $\operatorname{Div}(X) \to \operatorname{Div}(X_0)$ is the restriction map and $\operatorname{Div}(X'_0) \to \operatorname{Div}(X')$ is defined by taking the Zariski closure of divisors. Note that $\pi_*\pi^\star(D) = \deg(\pi)D$ for all $D \in \operatorname{Div}(X)$.

First, we assume (1). Note that the number of prime divisors in $X' \setminus X'_0$ is finite, so that it is sufficient to show that the number of prime divisors Γ' on X' with $\Gamma' \not\subseteq X' \setminus X'_0$ and

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1)) \cdots \widehat{c}_1(\pi^*(\overline{H}_1)) \mid \Gamma') \leq h'$$

is finite. By the projection formula,

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_1))\,|\,\Gamma') = \widehat{\deg}(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_1)\,|\,\pi_*(\Gamma')).$$

Thus, by (1), the number of $(\pi_*(\Gamma'))_{\mathrm{red}}$ is finite. On the other hand, the number of prime divisors in $\pi^{-1}(\pi_*(\Gamma)_{\mathrm{red}})$ is finite. Hence we get (2).

Next, we assume (2). Let Γ be a prime divisor on X with $\widehat{\deg}(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_1)\,|\,\Gamma) \leq h$. Then,

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_1))\,|\,\pi^\star(\Gamma)) = \deg(\pi)\widehat{\deg}(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_1)\,|\,\Gamma) \leq \deg(\pi)h.$$

Thus, by (2), the number of $\pi^\star(\Gamma)$'s is finite. Therefore, we get (1). □

Let us go back to the proof of Proposition 1.10.1. We use the notation in the above definition of strict finiteness. By Lemma 1.10.2, it is sufficient to show that the number of prime divisors Γ' on B' with

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma') \leq h$$

is finite for all h.

There are $\mathbb{Q}$-effective C^∞-hermitian line bundles $\overline{Q}_1,\ldots,\overline{Q}_d$ on B' with

$$\pi^*(\overline{H}_i) = \mu^*(p_i^*(\overline{L}_i)) \otimes \overline{Q}_i$$

for all i. Note that

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma') = \widehat{\deg}(\widehat{c}_1(\mu^*(p_1^*(\overline{L}_1)))\cdots\widehat{c}_1(\mu^*(p_d^*(\overline{L}_d)))\,|\,\Gamma') +$$
$$\sum_{i=1}^{d}\widehat{\deg}(\widehat{c}_1(\mu^*(p_1^*(\overline{L}_1)))\cdots\widehat{c}_1(\mu^*(p_{i-1}^*(\overline{L}_{i-1})))\cdot\widehat{c}_1(\overline{Q}_i)\cdot\widehat{c}_1(\pi^*(\overline{H}_{i+1}))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma').$$

Moreover, since $\overline{Q}_i$ is $\mathbb{Q}$-effective, the number of prime divisors Γ' with

$$\widehat{\deg}(\widehat{c}_1(\mu^*(p_1^*(\overline{L}_1)))\cdots\widehat{c}_1(\mu^*(p_{i-1}^*(\overline{L}_{i-1})))\cdot\widehat{c}_1(\overline{Q}_i)\cdot\widehat{c}_1(\pi^*(\overline{H}_{i+1}))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma') < 0$$

is finite for every i. Thus, we have

$$\widehat{\deg}(\widehat{c}_1(\pi^*(\overline{H}_1))\cdots\widehat{c}_1(\pi^*(\overline{H}_d))\,|\,\Gamma') \geq \widehat{\deg}(\widehat{c}_1(\mu^*(p_1^*(\overline{L}_1)))\cdots\widehat{c}_1(\mu^*(p_d^*(\overline{L}_d)))\,|\,\Gamma')$$

except finitely many Γ'. On the other hand, by [8, Proposition 4.1], the number of prime divisors Γ'' on $(\mathbb{P}^1_{\mathbb{Z}})^d$ with

$$\widehat{\deg}(\widehat{c}_1(p_1^*(\overline{L}_1))\cdots\widehat{c}_1(p_d^*(\overline{L}_d))\,|\,\Gamma'') \leq h$$

is finite. Thus, we get our proposition. □

Remark 1.10.3. Let X be a projective normal arithmetic variety of dimension n. Let $\overline{H}_1,\ldots,\overline{H}_{n-2}$ be nef C^∞-hermitian line bundles on X and $\overline{L}$ a C^∞-hermitian line bundle on X. If $\overline{L}$ is pseudo-effective, then we can expect the number of prime divisors Γ on X with

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_{n-2})\cdot\widehat{c}_1(\overline{L})\,|\,\Gamma) < 0$$

to be finite. If it is true, then Proposition 1.10.1 holds under the weaker assumption that the polarization is fine.

2. Height functions in terms of hermitian line bundles with logarithmic singularities

Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$. Let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a fine polarization of K. Let X be a projective variety over K and L an ample line bundle on X. Moreover, let Y be a proper closed subset of X. Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a pair of a projective arithmetic variety $\mathcal{X}$ and a hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ with the following properties:

(1) There is a morphism $f : \mathcal{X} \to B$ such that the generic fiber of f is X.
(2) $\mathcal{L}$ gives rise to L on the generic fiber of f.
(3) $\mathcal{L}$ is ample with respect to the morphism $f : \mathcal{X} \to B$.
(4) Let $\mathcal{Y}$ be a closed set of $\mathcal{X}$ such that $\mathcal{Y}$ gives rise to Y on the generic fiber of $\mathcal{X} \to B$. Then the hermitian metric of $\overline{\mathcal{L}}$ has logarithmic singularities along $\mathcal{Y}(\mathbb{C})$.

For $x \in X(\overline{K}) \setminus Y(\overline{K})$, we denote by Δ_x the Zariski closure of the image of $\operatorname{Spec}(\overline{K}) \to X \to \mathcal{X}$. The height of x with respect to $\overline{\mathcal{L}}$ is defined by

$$h_{\overline{\mathcal{L}}}(x) = \frac{\widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)\big|_{\Delta_x}) \cdots \widehat{c}_1(f^*(\overline{H}_d)\big|_{\Delta_x}) \cdot \widehat{c}_1(\overline{\mathcal{L}}\big|_{\Delta_x}))}{[K(x) : K]}.$$

Note that since $\overline{\mathcal{L}}\big|_{\Delta_x}$ has logarithmic singularities along $\mathcal{Y}(\mathbb{C}) \cap \Delta_x(\mathbb{C})$, the number

$$\widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)\big|_{\Delta_x}) \cdots \widehat{c}_1(f^*(\overline{H}_d)\big|_{\Delta_x}) \cdot \widehat{c}_1(\overline{\mathcal{L}}\big|_{\Delta_x}))$$

is well defined. Then, we have the following proposition.

Proposition 2.1. (1) *Let us fix a positive integer e. Then, there is a constant C such that*

$$\#\{x \in X(\overline{K}) \setminus Y(\overline{K}) \mid h_{\overline{\mathcal{L}}}(x) \leq h,\ [K(x) : K] \leq e\} \leq C \cdot h^{d+1}$$

for $h \gg 0$.
(2) *There is a constant C' such that $h_{\overline{\mathcal{L}}}(x) \geq C'$ for all $x \in X(\overline{K}) \setminus Y(\overline{K})$.*

Proof. We denote by $\|\cdot\|$ the hermitian metric of $\overline{\mathcal{L}}$. Let $\overline{Q}$ be an ample C^∞-hermitian line bundle on B. Then,

$$h_{\overline{\mathcal{L}} \otimes f^*(\overline{Q}^{\otimes n})}(x) = h_{\overline{\mathcal{L}}}(x) + n\widehat{\deg}(\widehat{c}_1(\overline{Q}) \cdot \widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d)).$$

Thus, we may assume that $\mathcal{L}$ is ample on $\mathcal{X}$. Moreover, replacing $\overline{\mathcal{L}}$ by $\overline{\mathcal{L}}^{\otimes n}$, we may assume that $\mathcal{I}_{\mathcal{Y}} \otimes \mathcal{L}$ is generated by global sections, where $\mathcal{I}_{\mathcal{Y}}$ is the defining ideal sheaf of $\mathcal{Y}$. Let $s_1, \ldots, s_r$ be generators of $H^0(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes \mathcal{L})$. We may view $s_1, \ldots, s_r$ as global sections of $H^0(\mathcal{X}, \mathcal{L})$. Then, $\mathcal{Y} = \{x \in \mathcal{X} \mid s_1(x) = \cdots = s_r(x) = 0\}$. Here we choose a C^∞-hermitian metric $\|\cdot\|_0$ of $\mathcal{L}$ such that $\|s_i\|_0 < 1/e$ for all $i = 1, \ldots, r$. We denote $(\mathcal{L}, \|\cdot\|_0)$ by $\overline{\mathcal{L}}^0$.

Here we claim

$$[K(x) : K] h_{\overline{\mathcal{L}}^0}(x) \geq -\int_{\Delta_x(\mathbb{C})} \log\left(\max_i\{\|s_i\|_0\}\right) c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)).$$

Indeed, we can find s_j with $s_j|_{\Delta_x} \neq 0$. Thus,

$$[K(x):K]h_{\overline{\mathcal{L}}^0}(x) = \widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \mid \operatorname{div}(s_j|_{\Delta_x})) - \int_{\Delta_x(\mathbb{C})} \log\left(\|s_j\|_0\right) c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)).$$

Hence, we get our claim because

$$\widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \mid \operatorname{div}(s_j|_{\Delta_x})) \geq 0 \quad \text{and} \quad \|s_j\|_0 \leq \max_i\{\|s_i\|_0\}.$$

Since $\|\cdot\|$ has logarithmic singularities, if we set $g = \|\cdot\|/\|\cdot\|_0$, then there is a positive constant a, b such that

$$|\log(g)| \leq a + b\log\left(-\log(\max_i\{\|s_i\|_0\})\right).$$

Moreover,

$$\left|h_{\overline{\mathcal{L}}}(x) - h_{\overline{\mathcal{L}}^0}(x)\right| \leq \frac{1}{[K(x):K]}\int_{\Delta_x(\mathbb{C})} |\log(g)| c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)).$$

Note that

$$\int_{\Delta_x(\mathbb{C})} c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)) = [K(x):K]\deg(\overline{B}),$$

where $\deg(\overline{B}) = \displaystyle\int_{B(\mathbb{C})} c_1(\overline{H}_1) \wedge \cdots \wedge c_1(\overline{H}_d)$ as in §§ 1.10. Thus,

$$\frac{\left|h_{\overline{\mathcal{L}}}(x) - h_{\overline{\mathcal{L}}^0}(x)\right|}{\deg(\overline{B})} \leq a + b\int_{\Delta_x(\mathbb{C})} \log\left(-\log(\max_i\{\|s_i\|_0\})\right) \frac{c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d))}{[K(x):K]\deg(\overline{B})}.$$

On the other hand,

$$\int_{\Delta_x(\mathbb{C})} \log\left(-\log(\max_i\{\|s_i\|_0\})\right) \frac{c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d))}{[K(x):K]\deg(\overline{B})} \leq \log\left(\int_{\Delta_x(\mathbb{C})} -\log(\max_i\{\|s_i\|_0\})\frac{c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d))}{[K(x):K]\deg(\overline{B})}\right).$$

Hence, we obtain

$$\frac{\left|h_{\overline{\mathcal{L}}}(x) - h_{\overline{\mathcal{L}}^0}(x)\right|}{\deg(\overline{B})} \leq a + b\log\left(\frac{h_{\overline{\mathcal{L}}^0}(x)}{\deg(\overline{B})}\right).$$

Note that there is a real number t_0 such that $a + b\log(t) \leq t/2$ for all $t \geq t_0$. Thus,

$$h_{\overline{\mathcal{L}}^0}(x) \leq \max\left\{\deg(\overline{B})t_0, 2h_{\overline{\mathcal{L}}}(x)\right\}.$$

Therefore, if $h \geq \deg(\overline{B})t_0/2$, then $h_{\overline{\mathcal{L}}}(x) \leq h$ implies $h_{\overline{\mathcal{L}}^0}(x) \leq 2h$. Hence, we get the first assertion by virtue of [8, Theorem 6.2.2].

Next let us see the second assertion. Since

$$\|s_i\| = g\|s_i\|_0 \leq \exp(a)\|s_i\|_0\left(-\log(\max_j\{\|s_j\|_0\})\right)^b \leq \exp(a)\|s_i\|_0\left(-\log(\|s_i\|_0)\right)^b$$

and the function $t(-\log(t))^b$ is bounded above for $0 < t \leq 1$, there is a constant C such that $\|s_i\| \leq C$ for all i. Thus, if we choose s_i with $s_i|_{\Delta_x} \neq 0$, then

$$\begin{aligned}
[K(x):K]h_{\overline{\mathcal{L}}}(x) &= \widehat{\deg}(\widehat{c}_1(f^*(\overline{H}_1)) \cdots \widehat{c}_1(f^*(\overline{H}_d)) \mid \operatorname{div}(s_i|_{\Delta_x})) \\
&\qquad - \int_{\Delta_x(\mathbb{C})} \log(\|s_j\|)\, c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)) \\
&\geq -\log(C) \int_{\Delta_x(\mathbb{C})} c_1(f^*(\overline{H}_1)) \wedge \cdots \wedge c_1(f^*(\overline{H}_d)) \\
&= -\log(C) \deg(\overline{B})[K(x):K].
\end{aligned}$$

Thus, we get (2). □

3. Faltings' modular height

Let K be a finitely generated field extension of $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$ and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a generically smooth polarization of K. Let A be a g-dimensional abelian variety over K. Let $\lambda(A/K; B)$ be the Hodge sheaf of A with respect to B (cf. §§ 1.6). Note that $\lambda(A/K; B)$ is invertible over $B_{\mathbb{Q}}$ because $B_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$. Let $\|\cdot\|_{\mathrm{Fal}}$ be Faltings' metric of $\lambda(A/K; B)$ over $B(\mathbb{C})$. Here we set

$$\overline{\lambda}^{\mathrm{Fal}}(A/K; B) = (\lambda(A/K; B), \|\cdot\|_{\mathrm{Fal}}),$$

which is called *the metrized Hodge sheaf of A with respect to B*. By Lemma 1.3.1, the metric of $\overline{\lambda}^{\mathrm{Fal}}(A/K; B)$ is locally integrable. Thus, *the Faltings' modular height of A with respect to the polarization $\overline{B}$* is defined by

$$h^{\overline{B}}_{\mathrm{Fal}}(A) = \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \cdot \widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K; B)\right).$$

Proposition 3.1. *Let $\pi : X' \to X$ be a generically finite morphism of normal projective generically smooth arithmetic varieties. Let K and K' be the function field of X and X' respectively. Let A be an abelian variety over K. Then, there is an effective divisor E on X with the following properties:*

(1) $\pi_* \widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A \times_K \operatorname{Spec}(K')/K'; X')) = \deg(\pi)\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K; X)) + (E, 0)$.

(2) *For a scheme S, we denote by $S^{(1)}$ the set of all codimension one points of S. Then,*

$$\{x \in X^{(1)} \mid A \textit{ has semi-abelian reduction at } x\} \subseteq (X \setminus \operatorname{Supp}(E))^{(1)}.$$

Moreover, if $A \times_K \operatorname{Spec}(K')$ has semi-abelian reduction in codimension one, then

$$\{x \in X^{(1)} \mid A \textit{ has semi-abelian reduction at } x\} = (X \setminus \operatorname{Supp}(E))^{(1)}.$$

Proof. (1) Let X_0 be the maximal Zariski open set of X such that X_0 is regular and π is finite over X_0. Then, $\operatorname{codim}(X \setminus X_0) \geq 2$. We set $X'_0 = \pi^{-1}(X_0)$ and $\pi_0 = \pi|_{X'_0}$. Let $\operatorname{Div}(X)$ and $\operatorname{Div}(X')$ be the groups of Weil divisors on X and X' respectively. Then, a homomorphism $\pi^\star : \operatorname{Div}(X) \to \operatorname{Div}(X')$ is defined by the compositions of homomorphisms:

$$\operatorname{Div}(X) \to \operatorname{Div}(X_0) \xrightarrow{\pi_0^*} \operatorname{Div}(X'_0) \to \operatorname{Div}(X'),$$

where $\mathrm{Div}(X) \to \mathrm{Div}(X_0)$ is the restriction map and $\mathrm{Div}(X'_0) \to \mathrm{Div}(X')$ is defined by taking the Zariski closure of divisors. Note that $\pi_*\pi^\star(D) = \deg(\pi)D$ for all $D \in \mathrm{Div}(X)$.

Let X_1 (resp. X'_1) be a Zariski open sets of X (resp. X') such that $\operatorname{codim}(X \setminus X_1) \geq 2$ (resp. $\operatorname{codim}(X' \setminus X'_1) \geq 2$) and the Néron model G (resp. G') exists over X_1 (resp. X'_1). Clearly we may assume that $X_1 \subseteq X_0$ and $\pi^{-1}(X_1) \subseteq X'_1$. We set $X'_2 = \pi^{-1}(X_1)$ and $G'_2 = G' \times_{X'_1} X'_2$. Since G'_2 is the Néron model of $A \times_K \mathrm{Spec}(K')$ over X'_2, there is a homomorphism $G \times_{X_1} X'_2 \to G'_2$ over X'_2. Thus, we get a homomorphism

$$\alpha : \pi^*\epsilon^* \left(\bigwedge^g \Omega_{G/X_1} \right) \to \epsilon'^* \left(\bigwedge^g \Omega_{G'_2/X'_2} \right), \tag{3.1.1}$$

where ϵ and ϵ' are the zero sections of G and G' respectively.

Let s be a non-zero rational section of $\lambda(A; X)$. Then,

$$\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K; X)) = (\mathrm{div}(s), -\log \|s\|_{\mathrm{Fal}}).$$

Moreover, since $\pi^*(s)$ gives rise to a non-zero rational section of $\lambda(A \times_K \mathrm{Spec}(K'); X')$,

$$\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A \times_K \mathrm{Spec}(K')/K'; X')) = (\mathrm{div}(\pi^*(s)), -\pi^*(\log \|s\|_{\mathrm{Fal}})),$$

where $\pi^*(\log \|s\|_{\mathrm{Fal}})$ is the pull-back of $\log \|s\|_{\mathrm{Fal}}$ by π as a function on a dense open set of $X(\mathbb{C})$. Let $\Gamma_1, \ldots, \Gamma_r$ be all prime divisors in $X' \setminus X'_2$. Note that $\pi_*(\Gamma_i) = 0$ for all i. Then, since (3.1.1) is injective, there is an effective divisor E' and integers $a_1, \ldots, a_r$ such that

$$\mathrm{div}(\pi^*(s)) = \pi^\star(\mathrm{div}(s)) + E' + \sum_{i=1}^r a_i \Gamma_i.$$

Note that $E' = \sum_{x'} \mathrm{length}_{\mathcal{O}_{X',x'}}(\mathrm{Coker}(\alpha)_{x'})\overline{\{x'\}}$, where x''s run over all codimension one points of X'_2. Thus, since $\pi_*(\pi^\star(\mathrm{div}(s)), -\pi^*(\log \|s\|_{\mathrm{Fal}})) = \deg(\pi)(\mathrm{div}(s), -\log \|s\|_{\mathrm{Fal}})$, we have

$$\pi_*\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A \times_K \mathrm{Spec}(K')/K'; X')) = \deg(\pi)\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K; X)) + (\pi_*(E'), 0).$$

Therefore, we get (1).

Next let us see (2). We assume that A has semi-abelian reduction at x. Then, there is a open set U such that $x \in U$ and $G^{\mathrm{o}}|_U$ is semi-abelian. Thus, $G^{\mathrm{o}}|_U \times_U \pi^{-1}(U)$ is semi-abelian. Hence $\left(G'|_{\pi^{-1}(U)} \right)^{\mathrm{o}}$ is isomorphic to $G^{\mathrm{o}}|_U \times_U \pi^{-1}(U)$. Thus $x \notin E_{\mathrm{red}}$. Conversely, we assume that $A \times_K \mathrm{Spec}(K')$ has semi-abelian reduction in codimension one and $x \notin E_{\mathrm{red}}$. Then, there is an open set $U \subset X_1$ such that $x \in U$ and the homomorphism

$$\alpha : \pi^*\epsilon^* \left(\bigwedge^g \Omega_{G/X_1} \right) \to \epsilon'^* \left(\bigwedge^g \Omega_{G'_2/X'_2} \right)$$

is an isomorphism over $\pi^{-1}(U)$, that is, so is $\pi^*\epsilon^* \left(\Omega_{G/X_1} \right) \to \epsilon'^* \left(\Omega_{G'_2/X'_2} \right)$ over $\pi^{-1}(U)$. Thus, $G^{\mathrm{o}} \times_{X_1} X'_2 \to (G'_2)^{\mathrm{o}}$ is an isomorphism over $\pi^{-1}(U)$. Therefore, G^{o} is semi-abelian over U. $\square$

Proposition 3.2. *Let $\phi : A \to A'$ be an isogeny of abelian varieties over K. Then*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A'/K;B)\right) - \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K;B)\right)$$
$$= \frac{1}{2}\log(\deg(\phi))\deg(\overline{B}) - \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\mid D_\phi\right),$$

where D_ϕ is an effective divisor given in §§ 1.6 and $\deg(\overline{B}) = \int_{B(\mathbb{C})} c_1(\overline{H}_1)\wedge\cdots\wedge c_1(\overline{H}_d)$ as in §§ 1.10.

Proof. This follows from the fact that $\overline{\lambda}^{\mathrm{Fal}}(A'/K;B)\otimes(\mathcal{O}_B(D_\phi),\deg(\phi)|\cdot|_{\mathrm{can}})$ is isometric to $\overline{\lambda}^{\mathrm{Fal}}(A/K;B)$. □

Proposition 3.3. *If an abelian variety A over K has semi-abelian reduction in codimension one over B. Then,*

$$\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K;B)\right) = \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A^\vee/K;B)\right),$$

where $A^\vee$ is the dual abelian variety of A.

Proof. Let $\phi : A \to A^\vee$ be an isogeny over K in terms of ample line bundle on A. Let $\phi^\vee : A \to A^\vee$ be the dual of ϕ. Then, by Proposition 3.2,

$$2\left(\widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A^\vee/K;B)\right) - \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\cdot\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A/K;B)\right)\right)$$
$$= \log(\deg(\phi))\deg(\overline{B}) - \widehat{\deg}\left(\widehat{c}_1(\overline{H}_1)\cdots\widehat{c}_1(\overline{H}_d)\mid D_\phi + D_{\phi^\vee}\right).$$

On the other hand, by Lemma 1.6.3, $\mathcal{I}_\phi\cdot\mathcal{I}_{\phi^\vee} = \deg(\phi)\mathcal{O}_B$. Thus, $(\mathcal{O}_B(D_\phi + D_{\phi^\vee}), |\cdot|_{\mathrm{can}})$ is isometric to $(\mathcal{O}_B, \deg(\phi)^{-2}|\cdot|_{\mathrm{can}})$. Therefore, we get our proposition. □

Let K be a finitely generated field extension of $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$ and $\overline{B} = (B;\overline{H}_1,\ldots,\overline{H}_d)$ a polarization of K. Let A be an abelian variety over a finite extension field K' of K. Let m be a positive integer such that m has a decomposition $m = m_1m_2$ with $(m_1, m_2) = 1$ and $m_1, m_2 \geq 3$. Let us consider a natural homomorphism

$$\rho(A,m) : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(A[m](\overline{K})) \simeq \mathrm{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}).$$

Then, there is a Galois extension $K(A,m)$ of K' with $\operatorname{Ker}\rho(A,m) = \mathrm{Gal}(\overline{K}/K(A,m))$. Note that

$$\mathrm{Gal}(K(A,m)/K') = \mathrm{Gal}(\overline{K}/K)/\operatorname{Ker}\rho(A,m) \hookrightarrow \mathrm{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}).$$

Let B'' be a generically smooth, normal and projective arithmetic variety with the following properties:

(i) The function field K'' of B'' is an extension of $K(A,m)$.
(ii) The natural rational map $f : B'' \to B$ induced by $K \hookrightarrow K''$ is actually a morphism.

Then, we have the following.

Proposition 3.4. (1) *The number*

$$\frac{1}{[K'':K]}\widehat{\deg}\left(\widehat{c}_1(\lambda(A\times_{K'}\operatorname{Spec}(K'')/K'';B'')\cdot\widehat{c}_1(f^*(\overline{H}_1))\cdots\widehat{c}_1(f^*(\overline{H}_1))\right)$$

does not depend on the choice of m and B'', so that we denote it by $h^{\overline{B}}_{\mathrm{mod}}(A)$.

(2) $h^{\overline{B}}_{\mathrm{mod}}(A)\le h^{\overline{B}}_{\mathrm{Fal}}(A)$.

Proof. These are consequences of Proposition 1.5.1, Proposition 3.1 and the projection formula. □

Proposition 3.5. *Let K be a finitely generated extension field of $\mathbb{Q}$. For abelian varieties A and A' over K, $h^{\overline{B}}_{\mathrm{Fal}}(A\times_K A')=h^{\overline{B}}_{\mathrm{Fal}}(A)+h^{\overline{B}}_{\mathrm{Fal}}(A')$. Moreover, $h^{\overline{B}}_{\mathrm{mod}}(A\times_K A')=h^{\overline{B}}_{\mathrm{mod}}(A)+h^{\overline{B}}_{\mathrm{mod}}(A')$.*

Proof. Let $\mathcal{A}$ and $\mathcal{A}'$ be the Néron models of A and A' over B_0, where B_0 is a big open set of B. Then, $\mathcal{A}\times_{B_0}\mathcal{A}'$ is the Néron model of $A\times_K A'$ over B_0. Thus,

$$\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{\mathcal{A}\times_{B_0}\mathcal{A}'/B})=\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{\mathcal{A}/B_0})+\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{\mathcal{A}'/B_0}).$$

Hence, we get our lemma. □

4. Weak finiteness

Let us fix positive integers g, l and m such that m has a decomposition $m=m_1m_2$ with $(m_1,m_2)=1$ and $m_1,m_2\ge 3$. Let $\mathbb{A}_{g,l,m,\mathbb{Q}}$, $f:Y\to\mathbb{A}^*_{g,l,m}$, $\overline{L}$, n and $G\to Y$ be the same as in Proposition 1.7.1.

Let K be a finitely generated field extension of $\mathbb{Q}$ with $d=\operatorname{tr.deg}_{\mathbb{Q}}(K)$ and let $\overline{B}=(B;\overline{H}_1,\ldots,\overline{H}_d)$ be a generically smooth polarization of K.

Let A be a g-dimensional and l-polarized abelian variety over a finite extension K' of K with an m-level structure. Let $x_A:\operatorname{Spec}(K')\to\mathbb{A}^*_{g,l,m}$ be the morphism induced by A. Moreover, let $y_A:\operatorname{Spec}(K')\to\mathbb{A}^*_{g,l,m}\times_{\mathbb{Z}}\operatorname{Spec}(K)$ be the morphism induced by x_A. Let Δ_A be the closure of the image of y_A in $\mathbb{A}^*_{g,l,m}\times_{\mathbb{Z}}B$. Let $p:\mathbb{A}^*_{g,l,m}\times_{\mathbb{Z}}B\to\mathbb{A}^*_{g,l,m}$ and $q:\mathbb{A}^*_{g,l,m}\times_{\mathbb{Z}}B\to B$ be the projections to the first factor and the second factor respectively. Here, we set

$$h^{\overline{B}}_{\overline{L}}(A)=\frac{1}{\deg(\Delta_A\to B)}\widehat{\deg}\left(\widehat{c}_1(q^*(\overline{H}_1)\big|_{\Delta_A})\cdots\widehat{c}_1(q^*(\overline{H}_d)\big|_{\Delta_A})\cdot\widehat{c}_1(p^*(\overline{L})\big|_{\Delta_A})\right)$$

which is nothing more than the height of $y_A\in(\mathbb{A}^*_{g,l,m}\times_{\mathbb{Z}}\operatorname{Spec}(K))(\overline{K})$ with respect to $\overline{L}$ and $\overline{B}$. Then, we have the following proposition.

Proposition 4.1. *There is a constant $N(g,l,m)$ depending only on g,l,m such that*

$$|h^{\overline{B}}_{\overline{L}}(A)-nh^{\overline{B}}_{\mathrm{mod}}(A)|\le\log(N(g,l,m))\deg(\overline{B}).$$

for every g-dimensional and l-polarized abelian variety A over $\overline{K}$ with an m-level structure, where

$$\deg(\overline{B})=\int_{B(\mathbb{C})}c_1(\overline{H}_1)\wedge\cdots c_1(\overline{H}_d).$$

Proof. Let A be a g-dimensional and l-polarized abelian variety over $\overline{K}$ with an m-level structure. Let K' be the minimal finite extension of K such that A, the polarization of A, the m-level structure of A are defined over K'. Let $x_A : \mathrm{Spec}(K') \to \mathbb{A}^*_{g,l,m}$ be the morphism induced by A. Moreover, let $y_A : \mathrm{Spec}(K') \to \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B$ be the induced morphism by x_A.

Let $\mathrm{Spec}(K_1)$ be a closed point of $Y \times_{\mathbb{A}^*_{g,l,m}} \mathrm{Spec}(K')$. Then, we have the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[d, "f"'] & \mathrm{Spec}(K_1) \arrow[l] \arrow[d] \\
\mathbb{A}^*_{g,l,m} & \mathrm{Spec}(K') \arrow[l, "x_A"']
\end{tikzcd}$$

Here, two l-polarized abelian varieties $A \times_{K'} \mathrm{Spec}(K_1)$ and $G \times_Y \mathrm{Spec}(K_1)$ with m-level structures gives rise to the same K_1-valued point of $\mathbb{A}^*_{g,l,m}$. Thus, $A \times_{K'} \mathrm{Spec}(K_1)$ is isomorphic to $G \times_Y \mathrm{Spec}(K_1)$ over K_1 as l-polarized abelian varieties with m-level structures because $m \geq 3$. The above commutative diagram gives rise to the commutative diagram:

$$\begin{tikzcd}
Y \times_{\mathbb{Z}} B \arrow[d] & \mathrm{Spec}(K_1) \arrow[l] \arrow[d] \\
\mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B & \mathrm{Spec}(K') \arrow[l, "y_A"']
\end{tikzcd}$$

Let B_1 be a generic resolution of singularities of the normalization of B in K_1. Note that a generic resolution of singularities (a resolution of singularities over $\mathbb{Q}$) exists by Hironaka's theorem [5]. Then, we have rational maps $B_1 \dashrightarrow Y \times_{\mathbb{Z}} B$ and $B_1 \dashrightarrow \Delta_A$ such that a composition $B_1 \dashrightarrow \Delta_A \to \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B$ of rational maps is equal to $B_1 \dashrightarrow Y \times_{\mathbb{Z}} B \to \mathbb{A}^*_{g,m} \times_{\mathbb{Z}} B$. Thus, there are a birational morphism $B_2 \to B_1$ of projective and generically smooth arithmetic varieties, a morphism $B_2 \to \Delta_A$ and a morphism $B_2 \to Y \times_{\mathbb{Z}} B$ with the following commutative diagram:

$$\begin{tikzcd}
B_1 \arrow[d, "\pi_1"'] & B_2 \arrow[l, "\gamma"'] \arrow[r, "\beta"] \arrow[d, "\alpha"] & Y \times_{\mathbb{Z}} B \arrow[d, "f \times \mathrm{id}"'] \\
B & \Delta_A \arrow[l] \arrow[r, "\iota"] & \mathbb{A}^*_{g,l,m} \times_{\mathbb{Z}} B
\end{tikzcd}$$

Then,

$$\begin{aligned}
h_{\overline{L}}^{\overline{B}}(A) &= \frac{\widehat{\deg}\left(\widehat{c}_1(\iota^*(p^*(\overline{L}))) \cdot \widehat{c}_1(\iota^*(q^*(\overline{H}_1))) \cdots \widehat{c}_1(\iota^*(q^*(\overline{H}_1)))\right)}{\deg(\Delta_A \to B)} \\
&= \frac{\widehat{\deg}\left(\widehat{c}_1(\alpha^*(\iota^*(p^*(\overline{L})))) \cdot \widehat{c}_1(\alpha^*(\iota^*(q^*(\overline{H}_1)))) \cdots \widehat{c}_1(\alpha^*(\iota^*(q^*(\overline{H}_1))))\right)}{\deg(B_2 \to B)} \\
&= \frac{\widehat{\deg}\left(\widehat{c}_1(\beta^*((f \times \mathrm{id})^*(p^*(\overline{L})))) \cdot \widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1))) \cdots \widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1)))\right)}{\deg(B_2 \to B)}.
\end{aligned}$$

On the other hand, since $f^*(L) = \lambda_{G/Y}^{\otimes n}$ over $Y \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$, there is an integer N depending only on g, l and m such that

$$N f^*(L) \subseteq \lambda_{G/Y}^{\otimes n} \subseteq (1/N) f^*(L)$$

on Y. Thus,

$$N\beta^*(f \times \mathrm{id})^*(L) \subseteq (\lambda_{G\times_{\mathbb{Z}}B/Y\times_{\mathbb{Z}}B})^{\otimes n} \subseteq (1/N)\beta^*(f \times \mathrm{id})^*(L).$$

Therefore,

$$\begin{aligned}
-\frac{\widehat{\deg}(\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1)))\cdots\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1)))\,|\,(N))}{\deg(B_2 \to B)} + h_{\overline{L}}^{\overline{B}}(A) \\
\leq \frac{n\widehat{\deg}\left(\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{G\times_Y B_2/B_2}))\cdot\widehat{c}_1(\gamma^*({\pi_1}^*(\overline{H}_1)))\cdots\widehat{c}_1(\gamma^*({\pi_1}^*(\overline{H}_1)))\right)}{\deg(B_2 \to B)} \\
\leq \frac{\widehat{\deg}(\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1)))\cdots\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1)))\,|\,(N))}{\deg(B_2 \to B)} + h_{\overline{L}}^{\overline{B}}(A).
\end{aligned}$$

Note that

$$\widehat{\deg}(\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1)))\cdots\widehat{c}_1(\gamma^*(\pi_1^*(\overline{H}_1)))\,|\,(N)) = \log(N)\deg(B_2 \to B)\deg(\overline{B}).$$

By Proposition 1.5.1, we can see that $A \times_{K'} \operatorname{Spec}(K_1)$ has semi-abelian reduction in codimension one over B_1. On the other hand, by Proposition 3.1,

$$\gamma_*(\widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}_{G\times_Y B_2/B_2})) = \widehat{c}_1(\overline{\lambda}^{\mathrm{Fal}}(A \times_{K'} \operatorname{Spec}(K_1)/K_1; B_1)).$$

Therefore, we get

$$|h_{\overline{L}}^{\overline{B}}(A) - n h_{\mathrm{mod}}^{\overline{B}}(A)| \leq \log(N)\deg(\overline{B}).$$

□

Corollary 4.2. *Let K be a finitely generated field extension of $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$ and $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a generically smooth and fine polarization of K. Let us fix a positive integer l. Then, we have the following:*

1. *There is a constant C such that $C \leq h_{\mathrm{mod}}^{\overline{B}}(A)$ for any l-polarized abelian variety A over $\overline{K}$.*
2. *Let us fix a positive integer e. Then, there is a constant C' such that the number of the set*

$$\left\{ A \times_{K'} \operatorname{Spec}(\overline{K}) \;\middle|\; \begin{array}{l} A \text{ is a } g\text{-dimensional and } l\text{-polarized abelian variety over} \\ \text{a finite extension } K' \text{ of } K \text{ with } [K' : K] \leq e \text{ and } h_{\mathrm{mod}}^{\overline{B}}(A) \leq h. \end{array} \right\} \Big/ \simeq_{\bar{K}}$$

is less than or equal to $C' \cdot h^{d+1}$ for $h \gg 0$.

Proof. Let us fix a positive number m such that m has a decomposition $m = m_1 m_2$ with $(m_1, m_2) = 1$ and $m_1, m_2 \geq 3$. Then, any l-polarized abelian variety over $\overline{K}$ has a m-level structure. Thus, (1) is a consequence of Proposition 2.1 and Proposition 4.1.

Let A be an l-polarized abelian variety over a finite extension K' of K. Let K'' be the minimal extension of K' such that $A[m](\overline{K}) \subseteq A(K'')$. Then, $[K'' : K'] \leq \#(\operatorname{Aut}(\mathbb{Z}/m\mathbb{Z})^{2g})$. Thus, by using Proposition 2.1 and Proposition 4.1, we get (2). □

5. Galois descent

Let A be a g-dimensional abelian variety over a field k. Let m be a positive integer prime to the characteristic of k. Note that an m-level structure α of A over a finite extension k' of k is an isomorphism $\alpha : (\mathbb{Z}/m\mathbb{Z})^{2g} \to A[m](k')$. If k' is a finite Galois extension over k, then we have a homomorphism

$$\epsilon(k'/k, A, \alpha) : \mathrm{Gal}(k'/k) \to \mathrm{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g})$$

given by $\epsilon(k'/k, A, \alpha)(\sigma) = \alpha^{-1} \cdot \sigma_A \cdot \alpha$, where

$$\sigma_A : A \times_k \mathrm{Spec}(k') \xrightarrow{\mathrm{id}_A \times (\sigma^{-1})^a} A \times_k \mathrm{Spec}(k')$$

is the natural morphism arising from σ. Note that $(\sigma \cdot \tau)_A = \sigma_A \cdot \tau_A$.

Lemma 5.1. *Let (A, ξ) and (A', ξ') be polarized abelian varieties over a field k. Let m be a positive integer prime to the characteristic of k. Let α and α' be m-level structures of A and A' respectively over a finite Galois extension k' of k. Let $\phi : (A, \xi) \times_k \mathrm{Spec}(k') \to (A', \xi') \times_k \mathrm{Spec}(k')$ be an isomorphism as polarized abelian varieties over k'. If $m \geq 3$, $\phi \cdot \alpha = \alpha'$ and $\epsilon(k'/k, A, \alpha) = \epsilon(k'/k, A', \alpha')$, then ϕ descents to an isomorphism $(A, \xi) \to (A', \xi')$ over k.*

Proof. For $\sigma \in \mathrm{Gal}(k'/k)$, let us consider a morphism

$$\phi_\sigma = \sigma_{A'}^{-1} \cdot \phi \cdot \sigma_A : A \times_k \mathrm{Spec}(k') \to A' \times_k \mathrm{Spec}(k').$$

First of all, ϕ_σ is a morphism over k'. We claim that $\phi_\sigma \cdot \alpha = \alpha'$. Indeed, since $\alpha^{-1} \cdot \sigma_A \alpha = \alpha'^{-1} \cdot \sigma_{A'} \cdot \alpha'$, we have

$$\phi_\sigma \cdot \alpha = \sigma_{A'}^{-1} \cdot \phi \cdot \alpha \cdot \alpha^{-1} \cdot \sigma_A \cdot \alpha = \sigma_{A'}^{-1} \cdot \alpha' \cdot \alpha'^{-1} \cdot \sigma_{A'} \cdot \alpha' = \alpha'.$$

Thus, ϕ_σ preserves the level structures of $A \times_k \mathrm{Spec}(k')$ and $A' \times_k \mathrm{Spec}(k')$. Hence, since $m \geq 3$ and $\phi_\sigma \cdot \phi^{-1}$ preserve the polarization ξ of A over k' (hence $(\phi_\sigma \cdot \phi^{-1})^N = \mathrm{id}$ for $N \gg 1$), by virtue of Serre's theorem, we have $\phi_\sigma = \phi$, that is,

$$\phi \cdot \sigma_A = \sigma_{A'} \cdot \phi$$

for all $\sigma \in \mathrm{Gal}(k'/k)$. Therefore, ϕ descents to an isomorphism $(A, \xi) \to (A', \xi')$ over k. □

Proposition 5.2. *Let B be an irreducible normal scheme such that B is of finite type over $\mathbb{Z}$. Let K be the local ring at the generic point of B. For a fixed g-dimensional polarized abelian variety (C, ξ_C) over $\overline{K}$, we set*

$$\mathcal{S} = \left\{ (A, \xi) \;\middle|\; \begin{array}{l} (A, \xi) \text{ is a polarized abelian variety over } K \text{ with } (A, \xi) \times_K \mathrm{Spec}(\overline{K}) \simeq (C, \xi_C) \\ \text{and } A \text{ has semi-abelian reduction over } B \text{ in codimension one.} \end{array} \right\}.$$

Then, the number of isomorphism classes in $\mathcal{S}$ is finite.

Proof. For $(A, \xi) \in \mathcal{S}$, let B_A be a big open set of B over which we have a semi-abelian extension $\mathcal{X}_A \to B_A$ of A. Moreover, let $BR(A)$ be the set of codimension one points x of B_A such that the fiber of $\mathcal{X}_A$ over x is not an abelian variety.

Claim 5.2.1. *For any $(A, \xi), (A', \xi') \in \mathcal{S}$, $BR(A) = BR(A')$.*

Since $A \times_K \operatorname{Spec}(\overline{K}) \simeq A' \times_K \operatorname{Spec}(\overline{K})$, there is a finite extension K' of K with $A \times_K \operatorname{Spec}(K') \simeq A' \times_K \operatorname{Spec}(K')$. Let $\pi : B' \to B$ be the normalization of B in K'. Then, $\mathcal{X}_A \times_{B_A} \pi^{-1}(B_A)$ is isomorphic to $\mathcal{X}_{A'} \times_{B_{A'}} \pi^{-1}(B_{A'})$ over $\pi^{-1}(B_A \cap B_{A'})$. Thus, $\pi^{-1}(BR(A)) = \pi^{-1}(BR(A'))$. Therefore, we get our claim.

Let us fix a positive integer $m \geq 3$ and $A_0 \in \mathcal{S}$. We set

$$U = B \setminus \left((B \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}/m\mathbb{Z})) \cup \operatorname{Sing}(B) \cup \bigcup_{x \in BR(A_0)} \overline{\{x\}} \right).$$

Then, U is regular and of finite type over $\mathbb{Z}$. The characteristic of the residue field of any point of U is prime to m. Moreover, by the above claim, if we set $U_A = U \cap B_A$ for $A \in \mathcal{S}$, then $\mathcal{X}_A$ is an abelian scheme over U_A and $\operatorname{codim}(U \setminus U_A) \geq 2$.

Claim 5.2.2. *There is a finite Galois extension K' of K such that for any $(A, \xi) \in \mathcal{S}$, all m-torsion points of A belong to $A(K')$.*

For $(A, \xi) \in \mathcal{S}$, let K_A be the finite extension of K obtaining by adding all m-torsion points of A to K. Let V_A be the normalization of U in K_A. Then, it is well-known that V_A is étale over U_A. Moreover, by virtue of the purity of branch loci (cf. SGA 1, Exposé X, Thérème 3.1), V_A is étale over U. Let M be the union of finite extension K' of K such that the normalization of U in K' is étale over U. Then, it is easy to see that M is a Galois extension of K. Since $K_A \subseteq M$, we have a continuous homomorphism

$$\rho_A : \operatorname{Gal}(M/K) \to \operatorname{Aut}(A[m](\overline{K})) \simeq \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g})$$

such that $\ker(\rho_A) = \operatorname{Gal}(M/K_A)$. Since $\operatorname{Gal}(M/K) = \pi_1(U)$, by [2, Hermite-Minkowski theorem in Chapter VI], we have only finitely many continuous homomorphisms

$$\rho : \operatorname{Gal}(M/K) \to \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}).$$

Thus, there are only finitely many Galois groups $\{\operatorname{Gal}(M/K_A)\}_{A \in \mathcal{S}}$. Therefore, $\{K_A\}_{A \in \mathcal{S}}$ is finite as a subfield of M. Thus, we get our claim.

Claim 5.2.3. *For any $(A, \xi), (A', \xi') \in \mathcal{S}$, $(A, \xi) \times_K \operatorname{Spec}(K') \simeq (A', \xi') \times_K \operatorname{Spec}(K')$.*

There is a finite Galois extension K'' of K' such that an isomorphism

$$\phi : (A, \xi) \times_K \operatorname{Spec}(K'') \to (A', \xi') \times_K \operatorname{Spec}(K'')$$

is given over K''. Let α be an m-level structure of A over K'' and $\alpha' = \phi \cdot \alpha$. Then, $\epsilon(K''/K', A \times_K \operatorname{Spec}(K'), \alpha) = \epsilon(K''/K', A' \times_K \operatorname{Spec}(K'), \alpha') = 1$ because all m-torsion points of A and A' are defined over K'. Thus, $A \times_K \operatorname{Spec}(K'') \to A' \times_K \operatorname{Spec}(K'')$ descents to an isomorphism $(A, \xi) \times_K \operatorname{Spec}(K') \to (A', \xi') \times_K \operatorname{Spec}(K')$ by Lemma 5.1.

Finally, let us see the number of isomorphism classes in $\mathcal{S}$ is finite. Let us fix $(A_0, \xi_0) \in \mathcal{S}$ and an m-level structure α_0 of A_0 over K'. Let $\phi_A : (A_0, \xi_0) \times_K \operatorname{Spec}(K') \to (A, \xi) \times_K \operatorname{Spec}(K')$ be an isomorphism over K'. We set $\alpha_A = \phi_A \cdot \alpha_0$ and $\phi^A_{A'} = \phi_{A'} \cdot \phi_A^{-1} : A \times_K \operatorname{Spec}(K') \to A' \times_K \operatorname{Spec}(K')$ for $(A, \xi), (A', \xi') \in \mathcal{S}$. Then, $\alpha_{A'} = \phi^A_{A'} \cdot \alpha_A$. Here let us consider a map

$$\gamma : \mathcal{S} \to \operatorname{Hom}(\operatorname{Gal}(K'/K), \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}))$$

given by $\gamma(A) = \epsilon(K'/K, A, \alpha_A)$. By Lemma 5.1, if $\gamma(A) = \gamma(A')$, then $(A, \xi) \simeq (A', \xi')$ over K. Moreover, $\operatorname{Hom}(\operatorname{Gal}(K'/K), \operatorname{Aut}((\mathbb{Z}/m\mathbb{Z})^{2g}))$ is a finite set. Therefore, we get our proposition. □

6. Strong finiteness

In this section, we give the proof of the main result of this note.

Theorem 6.1. *Let K be a finitely generated field over $\mathbb{Q}$ with $d = \operatorname{tr.deg}_{\mathbb{Q}}(K)$. Let $\overline{B} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ be a generically smooth and strictly fine polarization of K. Then, for any numbers c, the number of isomorphism classes of abelian varieties defined over K with $h^{\overline{B}}_{\mathrm{Fal}}(A) \le c$ is finite.*

Proof. Let us consider the following two sets:

$$\mathcal{S}_0(c) = \left\{(A, \xi) \mid (A, \xi) \text{ is a principally polarized abelian variety over } K \text{ with } h^{\overline{B}}_{\mathrm{mod}}(A) \le 8c\right\}$$

$$\mathcal{S}(c) = \left\{A \mid A \text{ is an abelian variety over } K \text{ with } h^{\overline{B}}_{\mathrm{Fal}}(A) \le c\right\}$$

Then, by Corollary 4.2, $\left\{(A, \xi) \times \operatorname{Spec}(\bar{K}) \mid (A, \xi) \in \mathcal{S}_0(c)\right\} /\simeq_{\bar{K}}$ is finite. By Zarhin's trick (cf. [10, Exposé VIII, Proposition 1]), for an abelian variety A over K, $(A \times A^\vee)^4$ is principally polarized. Moreover,

$$h^{\overline{B}}_{\mathrm{mod}}((A \times A^\vee)^4) = 8h^{\overline{B}}_{\mathrm{mod}}(A).$$

by Proposition 3.3 and Proposition 3.5. Thus, if $A \in \mathcal{S}(c)$, then $(A \times A^\vee)^4 \in \mathcal{S}_0(c)$. Here, the number of isomorphism classes of direct factors of $(A \times A^\vee)^4 \times_K \operatorname{Spec}(\bar{K})$ is finite (cf. [10, Exposé VIII, Proposition 2]). Thus, $\{A \times_K \operatorname{Spec}(\bar{K}) \mid A \in \mathcal{S}(c)\}/\simeq_{\bar{K}}$ is finite. In particular, there is a constant C such that $C \le h^{\overline{B}}_{\mathrm{mod}}(A)$ for all $A \in \mathcal{S}(c)$.

Let K_A be the minimal finite extension of K such that $A[12](\bar{K}) \subseteq A(K_A)$. Then, $[K_A : K] \le \#\operatorname{Aut}((\mathbb{Z}/12\mathbb{Z})^{2g})$. Let B_A be a generic resolution of singularities of the normalization of B in K_A. By Proposition 1.5.1, $A \times_K \operatorname{Spec}(K_A)$ has semi-abelian reduction in codimension one over B_A. Thus, by Proposition 3.1, there is an effective divisor E_A on B with

$$h^{\overline{B}}_{\mathrm{Fal}}(A) - h^{\overline{B}}_{\mathrm{mod}}(A) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \mid E_A)}{[K_A : K]}.$$

Here $h^{\overline{B}}_{\mathrm{mod}}(A) \ge C$ for all $A \in \mathcal{S}(c)$. Thus, we can find a constant C' such that

$$\widehat{\deg}(\widehat{c}_1(\overline{H}_1) \cdots \widehat{c}_1(\overline{H}_d) \mid E_A) \le C'$$

for all $A \in \mathcal{S}(c)$. Therefore, by virtue of Proposition 1.10.1, there is a reduced effective divisor D on B such that, for all $A \in \mathcal{S}(c)$, A has semi-abelian reduction in codimension one over $B \setminus D$. Hence, by Proposition 5.2, we have our assertion. □

Remark 6.2. If the problem in Remark 1.10.3 is true, then Theorem 6.1 holds even if the polarization $\overline{B}$ is fine.

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto, 606-8502, Japan

E-mail address: `moriwaki@math.kyoto-u.ac.jp`